MNYMO

August 22, 2026

Guts

The answer key is below.

p	Answer	p	Answer
1	-1980	15	$1+\sqrt{5}$
2	30	16	$\frac{7}{3}$
3	6	17	$\frac{21}{32}$
4	500	18	$\frac{56}{15}$
5	130	19	118
6	1014	20	25
7	2018	21	87
8	$\sqrt{2}-1$	22	$\frac{7}{54}$
9	$\frac{7}{16}$	23	250
10	4595	24	76
11	12	25	1743.37
12	20480	26	51277.00804003184
13	4907	27	896
14	8104	28	

1. An arithmetic sequence $a_1, a_2, a_3, \ldots$ satisfies $a_{20} = 26$ and $a_{26} = 20$. Compute a_{2026}.

   Proposed by: Samuel Kretzschmar

   Answer: $\boxed{-1980}$

   Solution: The common difference is $\frac{a_{26}-a_{20}}{26-20} = -1$. Thus, $a_{2026} = a_{26} - 2000 = \boxed{-1980}$.

2. At the ARML lunch, Jack has 12 apples and a deck of 60 cards. The deck has 50 cards worth 2 points and 10 cards worth 5 points. He plays the game "Apples to Apples" and randomly assigns a different card to each of his apples. Compute the expected total points of the cards assigned to Jack's apples.

   Proposed by: Jack Gao

   Answer: $\boxed{30}$

   Solution: By linearity of expectation, it's enough to compute the expected score of one apple and multiply by 12. There is a $\frac{5}{6}$ chance a given apple scores 2 points and a $\frac{1}{6}$ chance an apple scores 5 points. So the expected score of a single apple is $\frac{5}{6} \cdot 2 + \frac{1}{6} \cdot 5 = \frac{15}{6}$, so Jack's expected total score is $\frac{15}{6} \cdot 12 = \boxed{30}$.

3. Find the positive integer b such that $201_b = 243_5$.

   Proposed by: Jefferson Zhou

   Answer: $\boxed{6}$

   Solution: Write $2b^2 + 1 = 201_b = 243_5 = 2 \cdot 5^2 + 4 \cdot 5 + 3 = 73$, so $b = \sqrt{36} = \boxed{6}$.

4. In the town Xookland of 100 people, the GDW (Gross Domestic Water) initially starts at 0. Anytime someone receives ℓ liters of water, the GDW increases by ℓ. For example, if one person mysteriously receives 100 liters of water, then gives 50 liters to their neighbor, the GDW increases by 150.

   One morning, one of the people in the town mysteriously receives 100 liters of water. The following process repeats until everyone receives some water: the last person to receive water keeps $\frac{1}{5}$ of it and gives the remainder to another person who has not yet received any water. Afterwards, what will Xookland's GDW be? Round your answer to the nearest integer.

   Proposed by: Jefferson Zhou

   Answer: $\boxed{500}$

   Solution: The nth person to receive water gets $100 \cdot \left(\frac{4}{5}\right)^{n-1}$ of the water. Thus, the GDW after all 100 people get water is

   $$100 + 100 \cdot \frac{4}{5} + \cdots + 100 \cdot \left(\frac{4}{5}\right)^{99}.$$

   Since the latter terms are very small, we can approximate this as the geometric series $100 + 100 \cdot \frac{4}{5} + \cdots = \frac{100}{1-\frac{4}{5}} = \boxed{500}$.

   Remark. Most of the people to get water will likely get less than a drop of water, but in our perfectly idealized world, they still get nonzero water, so it stands.

5. Rectangle $ABCD$ has side lengths $AB = 28$ and $BC = 16$. Points E and F are constructed such that $ABCD$ is similar to $ABEF$ and E does not lie on CD. Compute the sum of all possible values of DF.

   Proposed by: Taison Scofield

   Answer: $\boxed{130}$

   Solution: Since $ABEF$ is similar to $ABCD$ with $AB = 28$, its other side can be either 16 or $\frac{28^2}{16} = 49$.

   Because EF can't lie on CD, the 16-unit version must extend to the opposite side as CD, making $DF = 32$. If $AF = 49$, DF can equal $16 + 49 = 65$ or $49 - 16 = 33$. In total, we get $32 + 65 + 33 = \boxed{130}$.

6. Every day for 2026 days, Trisha walks in a grove of 2026 trees, labeled Tree 1 to Tree 2026. On day N for $1 \leq N \leq 2026$, Trisha knocks on every Tree M for which M is a factor of N. At the end of the 2026th day, Tree K has not been knocked on for longer than any other tree. Find K.

   Proposed by: Samuel Kretzschmar

   Answer: $\boxed{1014}$

   Solution: Tree 1014 has not been knocked on since day 1014. Note that every positive integer $n \leq 1013$ has a multiple between 1014 and 2026 inclusive, so they have been knocked on after day 101, and tree k is always knocked on day k, so all trees with number greater than 1014 have been knocked on sooner. Thus the answer is $\boxed{1014}$.

7. The fraction $\frac{1}{2026!}$ can be written as a decimal $0.a_1a_2\dots a_k\overline{b_1b_2\dots b_n}$ using the $n+k$ digits $a_1, \dots, a_k, b_1, \dots, b_n$. What is the smallest possible value of k?

   Proposed by: Samuel Kretzschmar

   Answer: $\boxed{2018}$

   Solution: The answer is the minimum number k so that $\frac{10^k}{2026!}$ is a repeating decimal where the repeating block starts immediately after the decimal point; such a repeating decimal is obtained iff the denominator contains no factors of 2s or 5s. Clearly $\nu_2(2026!) \geq \nu_5(2026!)$, so we just need $\nu_2(2026!)$, which is

   $$1013 + 506 + 253 + 126 + 63 + 31 + 15 + 7 + 3 + 1 = \boxed{2018}$$

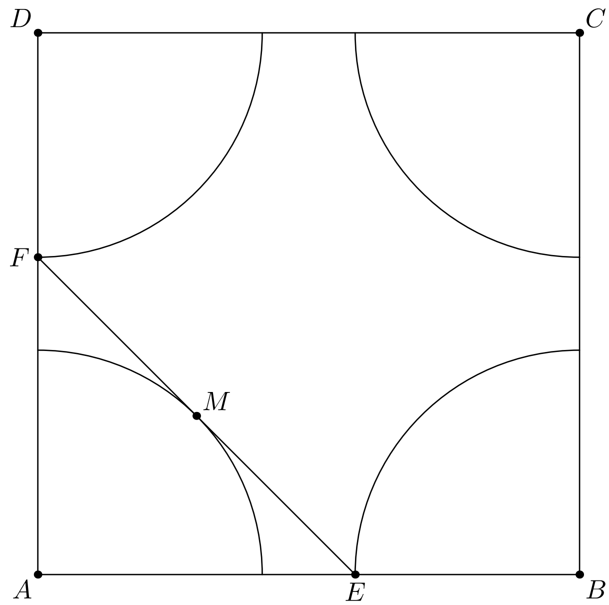
   by Legendre's theorem.

8. Jeff has a square piece of bread of side length 1. At each vertex, there is an air bubble in the shape of a quarter circle of radius r centered at that vertex. At two opposite corners of the bread, Jeff cuts out the smallest isosceles right triangle that contains the air bubble and has a vertex at the corner. After this, Jeff is surprised to learn that no part of the other two air bubbles has been cut out. Compute the maximum possible value of r.

   Proposed by: Jefferson Zhou

   Answer: $\boxed{\sqrt{2} - 1}$

   Solution:



Let the square be $ABCD$ and let one of the cut-out isosceles right triangles be AEF, as shown below. The situation at the opposite corner is identical, so we can ignore it. Let M be the midpoint of EF.

At the maximum value of r, E will lie at the point where the quarter circle centered at B intersects AB, and similarly for F. Thus $DF = BE = r$, so $AF = AE = 1 - r$. Because $\triangle AEF$ is the smallest isosceles right triangle with a vertex at A to fully contain the quarter circle at A, the quarter circle is tangent to EF at M. Thus, $AM = r = \frac{AF}{\sqrt{2}} = \frac{1-r}{\sqrt{2}}$. Now, $r(\sqrt{2}+1) = 1$, so $r = \frac{1}{\sqrt{2}+1} = \boxed{\sqrt{2}-1}$.

9. Michael has three fair four-sided dice, each of which has faces labeled 1, 2, 3, and 4. Compute the probability that the sum of the squares of the three rolls is divisible by 3.

   Proposed by: Angie Huang and Samuel Kretzschmar

   Answer: $\boxed{\frac{7}{16}}$

   Solution: The only perfect squares modulo 3 are 0 and 1, hence the only way for the sum of squares to be divisible by 3 is for all the numbers to be divisible by 3 or all the numbers to not be divisible by 3. This gives the probability

$$\left(\frac{3}{4}\right)^3 + \left(\frac{1}{4}\right)^3 = \frac{27}{64} + \frac{1}{64} = \boxed{\frac{7}{16}}.$$

10. Consider the sequence $1, 2, 1, 2, 2, 1, 2, 2, 2, 1, 2, 2, 2, 2, 1, \ldots$, where the number of 2's between consecutive 1's increases by 1 each time. If a_n represents the nth term, find the sum of all positive integers $n \leq 100$ such that $a_n = 2$.

    Proposed by: Taison Scofield

    Answer: $\boxed{4595}$

Solution: Notice that $a_n = 1$ when n is a triangular number and $a_n = 2$ otherwise. We use complementary counting. The sum of all positive integers $n \leq 100$ is

$$\sum_{n=1}^{100} n = \frac{100 \cdot 101}{2} = 5050.$$

The triangular numbers between 1 and 100 are $\binom{2}{2}$ through $\binom{14}{2}$, so by the hockey-stick identity, their sum is

$$\sum_{k=2}^{14} \binom{k}{2} = \binom{15}{3} = \frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1} = 455.$$

Therefore, the sum of all non-triangular numbers between 1 and 100 is $5050 - 455 = \boxed{4595}$.

11. Let $\triangle ABC$ be an equilateral triangle with a side length of 10. Let G be a point in the interior of $\triangle ABC$. Points D, E, and F are on line segments BC, AC, and AB, respectively, such that $GD \perp BC$, $GE \perp AC$, and $GF \perp AB$. If $GD : GE : GF = 1 : 2 : 2$, compute $AF + AE$.

    Proposed by: Jack Gao

    Answer: $\boxed{12}$

    Solution: By Viviani's Theorem, we know $GD + GE + GF = h$ where h is the altitude of $\triangle ABC$. Using the Pythagorean Theorem, $h = \sqrt{10^2 - 5^2} = 5\sqrt{3}$. So, $GE = GF = 2\sqrt{3}$. Using simple coordinate geometry, where $A = (0, 5\sqrt{3}), B = (-5, 0)$, and $C = (5, 0)$, point G is $(0, \sqrt{3})$. So, $AG = 4\sqrt{3}$. Using the Pythagorean Theorem again, we can see that $AF = AE = \sqrt{AG^2 - (2\sqrt{3})^2} = 6$. Therefore, $AF + AE = \boxed{12}$.

    Alternative Finish: Because $GF = GE$, G lies on the angle bisector of $\angle BAC$, which is also the altitude; thus, $AG = 5\sqrt{3} - \sqrt{3} = 4\sqrt{3}$. Thus $\triangle AFG$ and $\triangle AEG$ are actually 30–60–90 triangles with $AF = AE = 6$, so $AE + AF = \boxed{12}$.

12. In the game 2048, tiles with the same number can be merged to form a tile with twice the value. Each time two tiles merge, your score increases by the value of the newly created tile. Suppose all new tiles that appear are 2s and your score starts at 0. You have just created a 2048 tile, and immediately afterward a new 2 tile appears. If the board now contains only these two tiles, what is your score?

    Proposed by: Taison Scofield

    Answer: $\boxed{20480}$

    Solution: To obtain a 2048 tile, the following tiles must be created first:

    - 2 tiles of value 1024,
    - $2 \cdot 2 = 4$ tiles of value 512,
    - $4 \cdot 2 = 8$ tiles of value 256,
    - $8 \cdot 2 = 16$ tiles of value 128, ...
    - and so on until 512 tiles of value 4.

In general, we need 2^n tiles of value 2^{11-n} from $n = 0$ to $n = 9$. Notice that $2^n \cdot 2^{11-n} = 2^{11}$. Therefore, each merge level contributes $2^{11} = 2048$ to the final score. Since there are 10 merge levels, our final score is $\boxed{20480}$.

13. Find the unique four-digit integer which is 6 less than a cube and 7 more than a square.

    Proposed by: Samuel Kretzschmar

    Answer: $\boxed{4907}$

    Solution: Trying all possible values for the cube, we see that

    - $1331 - 13 = 1318$ is not a perfect square because its last digit is 8.
    - $1728 - 13 = 1715$ is not a perfect square because it's divisible by 5 but not 25.
    - $2197 - 13 = 2184$ is not a perfect square because it's $21 \cdot 26 \cdot 4$.
    - $2744 - 13 = 2731$ is not a perfect square because it's 3 modulo 4.
    - $3375 - 13 = 3362$ is not a perfect square because it ends in a 2.
    - $4096 - 13 = 4083$ is not a perfect square because it ends in a 3.
    - $4913 - 13 = 4900$ is a perfect square so $\boxed{4907}$ works. A contestant could just stop here.
    - $5832 - 13 = 5819$ is not a perfect square because it is 3 modulo 4.
    - $6859 - 13 = 6846$ is not a perfect square because it's 2 modulo 4.
    - $8000 - 13 = 7987$ is not a perfect square because it ends in a 7.
    - $9261 - 13 = 9248$ is not a perfect square because it ends in an 8.

    All further cubes are too big.

14. Sam plays a game on a five-by-five grid that initially has some positive number of coins in the center square. On each turn, he can take any two coins (possibly on different squares) and move them in opposite directions. However, Sam is not allowed to do any moves that would move coins outside the grid. Later in the game, he notices that if he draws a line between the top-left and bottom-right corners of the grid, exactly 2026 coins would be above this line. What is the maximum number of coins below this line?

    Proposed by: Zack Berchenko

    Answer: $\boxed{8104}$

    Solution: Let $N = 2026$; the answer is $4N = \boxed{8104}$.

    Impose coordinates on the grid such that the center is $(0, 0)$ and the corners are $(\pm 2, \pm 2)$. Let coin i have position (x_i, y_i). We require there to be N different i such that $x_i + y_i > 0$. We want to maximize the number of coins such that $x_i + y_i < 0$. Note that

    $$\sum_{\text{coin } i} (x_i + y_i)$$

    is invariant. Since the initial configuration has this sum equal to zero (as all coins start at $(0, 0)$), we know

    $$\sum_{\text{coin } i} (x_i + y_i) = 0.$$

    Let A be the set of coins above the line, and B be the set of coins below. Since $x_i + y_i \leq 4$ for all coins,

    $$\sum_{i \in A} (x_i + y_i) \leq 4N.$$

Furthermore, since all coins on the line have $x_i + y_i = 0$,

$$\sum_{i \in A}(x_i + y_i) + \sum_{i \in B}(x_i + y_i) = \sum_{\text{coin } i}(x_i + y_i) = 0.$$

Thus,

$$\sum_{i \in B}(x_i + y_i) \geq -4N.$$

Since $x_i + y_i \leq -1$, we have

$$|B| \cdot (-1) \geq \sum_{i \in B}(x_i + y_i) \geq -4N.$$

Therefore, $|B| \leq 4N$, so $4N$ is the maximum number of coins below the line. To obtain this maximum, Sam could first move N center coins up twice, moving N center coins down along the way. Then, Sam could move these N center coins right twice, moving another $2N$ center coins left along the way. This results in N coins above the line, and $2N + 2N = 4N$ coins below the line.

15. If x and y are real and $\log_y x + \log_x y = \log_2 x + \log_2 y = \log_x 4 + \log_y 4 = C$, compute C.

    Proposed by: Jefferson Zhou

    Answer: $\boxed{1+\sqrt{5}}$

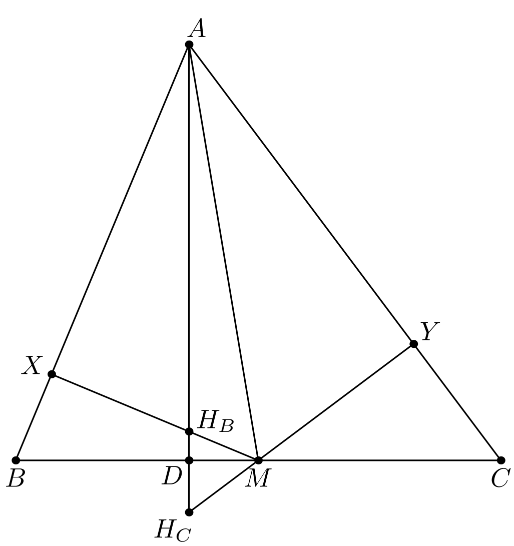
    Solution: Let $x = 2^a$ and $y = 2^b$. Then, the equations turn into $\frac{a}{b} + \frac{b}{a} = a + b = \frac{2}{a} + \frac{2}{b} = \frac{2(a+b)}{ab}$, so $ab = 2$. Now, $a + b = \frac{a}{b} + \frac{b}{a} = \frac{a^2+b^2}{ab} = \frac{(a+b)^2-2ab}{ab} = \frac{(a+b)^2-4}{2}$, so $(a+b)^2 - 2(a+b) - 4 = 0$. Now, $(a+b-1)^2 = (a+b)^2 - 2(a+b) + 1 = 5$, so $a + b = 1 \pm \sqrt{5}$. However, observe that $(a+b)^2 \geq 4ab = 8$, so $a + b \neq 1 - \sqrt{5}$, and thus $a + b = C = \boxed{1+\sqrt{5}}$.

16. Suppose $\triangle ABC$ has side lengths $AB = 13$, $BC = 14$, and $CA = 15$. If M is the midpoint of BC, H_B is the orthocenter of $\triangle BAM$, and H_C is the orthocenter of $\triangle CAM$, find the length of H_BH_C.

    Proposed by: Samuel Kretzschmar

    Answer: $\boxed{\frac{7}{3}}$

    Solution:



The key is to show that $\triangle MH_BH_C \sim \triangle ABC$. Let X and Y be the feet of the altitudes from M to AB and AC respectively, and D be the foot of the altitude from A to BC, as seen above. Then

$$\angle MH_BH_C = \angle XH_BA = 90^\circ - \angle XAD = \angle ABC$$

and

$$\angle MH_CH_B = \angle YH_CA = 90^\circ - \angle YAD = \angle ACB$$

so by AA similarity we have $\triangle MH_BH_C \sim \triangle ABC$. As $\triangle ABC$ is 13–14–15, it is well-known that $AD = 12$, $BD = 5$, and $MD = BM - BD = 7 - 5 = 2$. Noting that AD for $\triangle ABC$ corresponds to MD for $\triangle MH_BH_C$, by similarity

$$\frac{H_BH_C}{BC} = \frac{MD}{AD} \implies H_BH_C = \frac{14 \cdot 2}{12} = \boxed{\frac{7}{3}}.$$

17. Sinner and Alcaraz are playing American ping-pong. Each player has a $\frac{1}{2}$ chance of winning each point, and each point is independent. The players play points until one of the players has at least four points and is beating the other player by at least two points, at which point that player wins the game.

Alcaraz wins the first point to make the score 1–0. Compute the probability that Alcaraz wins the game.

Proposed by: Taison Scofield

Answer: $\boxed{\frac{21}{32}}$

Solution: We proceed by cases.

a) If Alcaraz can win without reaching 3–3, he needs to win 3 points before Sinner can win 3 points. Since each point is equally likely to be won by either player and both players need the same number of points, the probability that Alcaraz wins before deuce is $\frac{1}{2}$.

b) If the game reaches 3–3, Alcaraz needs to win exactly 2 of the next 5 points. The total number of outcomes for the next 5 points is $2^5 = 32$, while the number of ways for Alcaraz to win exactly 2 of the next 5 points is $\binom{5}{2} = 10$. Therefore, the chance that the game reaches 3–3 is $\frac{10}{32} = \frac{5}{16}$. From here, each player has a $\frac{1}{2}$ chance of winning, so the overall probability that Alcaraz wins via 3–3 is $\frac{5}{16} \cdot \frac{1}{2} = \frac{5}{32}$.

Adding up these two probabilities, we get $\frac{16}{32} + \frac{5}{32} = \boxed{\frac{21}{32}}$.

Alternative Solution: There's a $\frac{1}{2}$ chance the score is 1–1 after one point, in which case Alcaraz wins with probability $\frac{1}{2}$.

In the other case (also with probability $\frac{1}{2}$), it's 2–0. If it becomes 2–1, then there is a $\frac{1}{2}$ chance to become 2–2, which gives a winning probability of $\frac{1}{2}$ again. If it becomes 3–1, there is a $\frac{1}{2}$ chance of directly winning, a $\frac{1}{2} \cdot \frac{1}{2}$ of winning at 4–2, and a $\frac{1}{2} \cdot \frac{1}{2}$ chance of going to 3–3, where the probability of winning is $\frac{1}{2}$. Thus the probability of winning at 3–1 is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$, so the probability of winning at 2–1 is $\frac{1}{4} + \frac{1}{2} \cdot \frac{7}{8} = \frac{11}{16}$.

Similarly, if from 2–0, it goes to 3–0, there is a $\frac{1}{2}$ chance of directly winning and a $\frac{1}{2}$ chance of going to 3–1, so the total probability of winning is $\frac{1}{2} + \frac{1}{2} \cdot \frac{7}{8} = \frac{15}{16}$.

Thus the probability of winning from 2–0 is $\frac{1}{2} \cdot \frac{11}{16} + \frac{1}{2} \cdot \frac{15}{16} = \frac{13}{16}$, meaning the probability of winning from the start is $\frac{1}{4} + \frac{1}{2} \cdot \frac{13}{16} = \boxed{\frac{21}{32}}$.

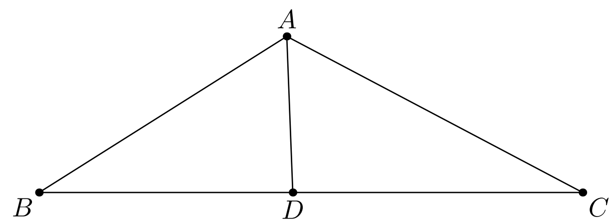
18. Find the unique positive value of x for which

$$\sqrt{x^2 - 7x + 49} + \sqrt{x^2 - 8x + 64} = 13.$$

Proposed by: Samuel Kretzschmar

Answer: $\boxed{\frac{56}{15}}$

Solution:



Notice that $\sqrt{x^2 - 7x + 49}$ and $\sqrt{x^2 - 8x + 64}$ can be represented using the Law of Cosines. Using x and 7 as side lengths and 60° as the angle, we get the third side length to be $\sqrt{x^2 + 49 - 2(x)(7)(\frac{1}{2})} = \sqrt{x^2 - 7x + 49}$. Let that be L_1.

Similarly, using x and 8 as side lengths and 60° as the angle, we get the third side length to be $\sqrt{x^2 + 64 - 2 \cdot x \cdot 8 \cdot \frac{1}{2}} = \sqrt{x^2 - 8x + 64}$. Let that be L_2.

Since both triangles share a common side of x, they can be combined into a quadrilateral with side lengths 7, 8, L_2, and L_1 and a diagonal of length x. We can find the length of the second diagonal using LoC since we know the angle between the sides of length 7 and 8 is $60 + 60 = 120^\circ$:

$$(D_2)^2 = 7^2 + 8^2 - 2 \cdot 7 \cdot 8 \cdot \cos(120^\circ) = 169 \implies D_2 = 13.$$

Since we were given in the problem statement that $L_1 + L_2 = 13$, L_1 and L_2 actually form D_2. That means instead of a quadrilateral, we have a triangle with lengths $7, 8,$ and 13. Let $AB = 7, BC = 13$, and $CA = 8$.

Notice that the line with length x forms an angle bisector of $\angle CAB$. Let the foot of the bisector (which lies on BC) be point D. By the Angle Bisector Theorem, we get

$$\frac{8}{CD} = \frac{7}{DB}$$

Using the fact that $CD + DB = 13$ yields $CD = \frac{104}{15}, DB = \frac{91}{15}$.

We can finish using the law of sines. As $\frac{AD}{\sin(\angle C)} = \frac{BC}{\sin(\angle A)}$,

$$x = \frac{104}{15} \cdot \frac{\sin(\angle C)}{\sin(60^\circ)}.$$

Since

$$\frac{\sin(\angle C)}{7} = \frac{\sin(\angle CAB)}{13} \implies \sin(\angle C) = 7\left(\frac{\sqrt{3}/2}{13}\right) \implies \sin(\angle C) = \frac{7\sqrt{3}}{26}$$

Plugging that into the previous equation yields $x = \frac{104}{15} \cdot \frac{7\sqrt{3}/26}{\sqrt{3}/2} = \frac{104}{15} \cdot \frac{7}{13} = \boxed{\frac{56}{15}}$

Alternative Solution: Let $a = x^2 - 7x + 49$, so the equation becomes

$$\sqrt{a} + \sqrt{a + 15 - x} = 13,$$

so $\sqrt{a + 15 - x} = 13 - \sqrt{a}$. Squaring both sides yields $a + 15 - x = 169 + a - 26\sqrt{a}$, so $26\sqrt{a} = x + 154$.

Now, $(x+154)^2 = x^2+308x+154^2 = 676a = 676x^2-4732x+182^2$, so $675x^2-5040x+28\cdot 336 = 0$. Dividing by 3, this becomes $0 = 225x^2 - 1680x + 28 \cdot 112 = (15x - 56)^2$, so $x = \boxed{\frac{56}{15}}$.

19. Find the largest possible value of $\gcd(n^3 + 3, n^5 + 5)$ across all positive integers n.

    Proposed by: Samuel Kretzschmar

    Answer: $\boxed{118}$

    Solution: Firstly, $\gcd(n^3+3, n^5+5) = \gcd(n^3+3, n^2(n^3+3)-(n^5+5)) = \gcd(n^3+3, 3n^2-5)$. Note that $3 \nmid 3n^2-5$, so this equals $\gcd(3n^3+9, 3n^2-5) = \gcd(3n^3+9-n(3n^2-5), 3n^2-5) = \gcd(5n+9, 3n^2-5)$.

    Because $5 \nmid 5n+9$, this equals

$$\begin{aligned} \gcd(5n+9, 15n^2-25) &= \gcd(5n+9, 3n(5n+9) - (15n^2-25)) \\ &= \gcd(5n+9, 27n+25) \\ &= \gcd(5n+9, 2n-20) \\ &= \gcd(n+49, 2n-20) \\ &= \gcd(n+49, 118), \end{aligned}$$

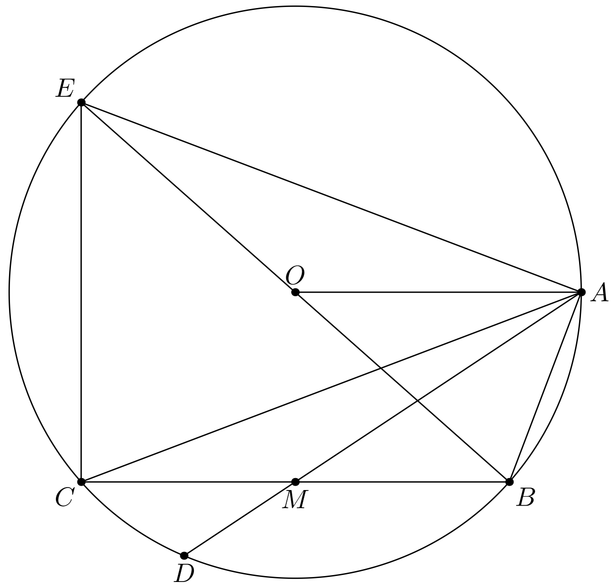
    so the maximum gcd equals $\boxed{118}$, attained at e.g. $n = 118 - 49 = 69$.

20. In triangle ABC, $\angle ABC = 90^\circ + \angle ACB$, point M is the midpoint of BC, and D is the intersection of AM with the circumcircle of ABC. If the circumradius of ABC is 15 and $AM = 18$, then what is AD?

    Proposed by: Jefferson Zhou

    Answer: $\boxed{25}$

    Solution:



Let E be the antipode of B on the circumcircle of ABC. Observe that $\angle ACE = \angle ABE = \angle ABO = \frac{180^\circ - 2C}{2} = 90^\circ - C = 180^\circ - \angle ABC = \angle AEC$, so $AE = AC$. Thus, $900 = BE^2 = AE^2 + AB^2 = AB^2 + AC^2$ by the Pythagorean Theorem.

By Stewart's theorem on $\triangle ABC$, we have

$$AB^2 + AC^2 = 2BM^2 + 2AM^2 = 648 + 2BM^2,$$

so $BM^2 = \frac{900-648}{2} = 126$. However, by Power of a Point, $126 = BM^2 = BM \cdot CM = AM \cdot DM = 18DM$, so $DM = 7$. Thus $AD = AM + MD = \boxed{25}$.

Alternative Solution: By the extended law of sines, $\frac{AC}{\sin B} = \frac{AC}{\cos C} = 30$ and $\frac{AB}{\sin C} = 30$, so $AB^2 + AC^2 = 30^2(\sin^2 C + \cos^2 C) = 900$. Finish identically to above.

21. A baker has a cookie in the shape of a regular pentagon. To decorate it, the baker makes n random cuts across the cookie, one at a time, as follows. The baker chooses two distinct sides of the cookie uniformly at random among the 10 possible pairs, then picks two points chosen independently and uniformly at random on these sides. The baker makes the cut connecting these two points.

Let n be the least integer greater than 1 such that the expected number of pieces into which the cookie is cut is an integer, and let E be that expected number of pieces. Find $n + E$.

Proposed by: Taison Scofield

Answer: $\boxed{87}$

Solution: We may assume that no two cuts share an endpoint and no three cuts pass through a common point, since these happen with probability 0.

First, notice that if we draw n chords inside a convex polygon, each new chord we add increases the number of regions by 1 plus the number of existing chords it crosses. That means if I is the total number of interior crossing points among all n chords, the number of regions is $1 + n + I$.

By linearity of expectation, $\mathbb{E}[\text{regions}] = 1 + n + \binom{n}{2}p$, where p is the probability that two independently drawn random chords cross. Since every pair of our 10 chords is drawn the same way, p is the same for every pair, so this reduces to finding p for a single pair of chords (AB and CD).

We can unroll the pentagon's boundary into a circle of circumference 5, letting side k be $[k, k+1)$. Two chords cross iff their four endpoints alternate around the circle.

WLOG fix chord AB. The two sides AB lands on are either right next to each other (distance 1) or have exactly one side between them (distance 2). Each of these cases happens 5 times out of the 10 possible side pairs, which is a $\frac{1}{2}$ probability.

For CD, the crossing probability just comes down to how many sides it shares with AB. If it shares both sides, the probability that two random chords living on the same two arcs intersect is $\frac{1}{2}$ by symmetry. If it shares exactly one side, that shared point is equally likely to land before or after A or B on its side, and the other endpoint's side is already fixed on one side or the other, so the probability is again $\frac{1}{2}$. If it shares no sides, we have two more cases depending on whether A and B are adjacent.

Case 1: A and B are on adjacent sides. The short arc between A and B is empty, so all 3 other sides sit in the long arc together. That means no "shares nothing" pair of CD can straddle the two arcs — they're all on the same side, so that case always contributes 0. Out of the 10 choices of CD: 1 shares both sides, 6 share exactly one, and 3 share none. Averaging, we get

$$p_1 = \frac{7 \cdot \frac{1}{2} + 3 \cdot 0}{10} = \frac{7}{20}.$$

Case 2: A and B are not on adjacent sides. Now the short arc has exactly 1 side in it, and the long arc has the other 2. So a "shares nothing" CD has its two points on opposite arcs (which guarantees a crossing) exactly when it picks the lone side and one of the two long-arc sides — that's $1 \times 2 = 2$ out of the 3 such pairs, each crossing with probability 1, leaving 1 pair that doesn't cross at all. Same breakdown otherwise (1 shares both, 6 share one). Averaging gives

$$p_2 = \frac{7 \cdot \frac{1}{2} + 2 \cdot 1 + 1 \cdot 0}{10} = \frac{11}{20}.$$

Since both cases occur with equal probability, we average these two to get the final value for p:

$$p = \frac{1}{2} \cdot \frac{7}{20} + \frac{1}{2} \cdot \frac{11}{20} = \frac{9}{20}.$$

This means

$$\mathbb{E}[\text{regions}] = 1 + n + \binom{n}{2} \cdot \frac{9}{20} = 1 + n + \frac{9n(n-1)}{40}.$$

Since $\gcd(9, 40) = 1$, this is an integer iff $40 \mid n(n-1)$. Hence we need $8 \mid n(n-1)$ and $5 \mid n(n-1)$. Since n and $n-1$ are coprime, this splits as independent conditions modulo 5 and 8. We find that $n(n-1) \equiv 0 \pmod 8$ when $n \equiv 0, 1 \pmod 8$, and $n(n-1) \equiv 0 \pmod 5$ when $n \equiv 0, 1 \pmod 5$. By the Chinese remainder theorem, the solutions mod 40 are

$$n \equiv 0, 1, 16, 25 \pmod{40}.$$

The least $n > 1$ satisfying this is $n = 16$. Plugging in $n = 16$, the expected number of regions is

$$E = 1 + 16 + \binom{16}{2} \cdot \frac{9}{20} = 17 + 120 \cdot \frac{9}{20} = 17 + 54 = 71.$$

Thus, $n + E = 16 + 71 = \boxed{87}$.

22. Real numbers b and c satisfy the system of equations

$$\frac{9b^3+9c^3+1}{15} = b^2 - bc + c^2$$
$$(b-2c)^2 + (c-2b)^2 = \frac{2}{9}.$$

Compute the sum of all the possible values of bc.

Proposed by: Jefferson Zhou

Answer: $\boxed{\frac{7}{54}}$

Solution: The bottom equation simplifies to $5b^2 + 5c^2 = 8bc + \frac{2}{9}$, so

$$9b^3 + 9c^3 + 1 = 15(b^2 - bc + c^2) = 15(b^2 + c^2) - 15bc = 24bc + \frac{2}{3} - 15bc = 9bc + \frac{2}{3}.$$

Now, $\frac{1}{3} + 9b^3 + 9c^3 - 9bc = 0$. Dividing by 9, we obtain

$$\frac{1}{27} + b^3 + c^3 - bc = \left(\frac{1}{3} + b + c\right)\left(\frac{(\frac{1}{3} - b)^2 + (\frac{1}{3} - c)^2 + (b-c)^2}{2}\right) = 0.$$

Thus, we have two cases: $b + c = -\frac{1}{3}$, or $b = c = \frac{1}{3}$. Observe that $b = c = \frac{1}{3}$ satisfies the equations, so $\frac{1}{9}$ is a possible value of bc.

Now, if $b + c = -\frac{1}{3}$, we have

$$\frac{2}{9} = 5b^2 + 5c^2 - 8bc = 5(b+c)^2 - 18bc = \frac{5}{9} - 18bc,$$

so $18bc = \frac{1}{3}$ and $bc = \frac{1}{54}$. It is easy to check that this yields real solutions, so $\frac{1}{54}$ is a possible value of bc.

Now, the answer is $\frac{1}{9} + \frac{1}{54} = \boxed{\frac{7}{54}}$.

Alternative Solution: Substitute $\sigma = b + c$ and $\pi = bc$; since b and c are real, we require $\sigma^2 \geq 4\pi$. The key observation is that both the first and second equations are linear in π, hence we can just solve for it in terms of σ: we have

$$\pi = \frac{9\sigma^3 - 15\sigma^2 + 1}{27\sigma - 45} = \frac{5\sigma^2 - \frac{2}{9}}{18}.$$

Using our second expression for π, the condition $\sigma^2 \geq 4\pi$ becomes $\sigma^2 \leq \frac{4}{9}$. Equating our two expressions for π yields the cubic

$$27\sigma^3 - 45\sigma^2 + 6\sigma + 8 = 0.$$

Letting $\sigma' = 3\sigma$, the cubic easily factors as $(\sigma' - 4)(\sigma' - 2)(\sigma' + 1) = 0$, so the valid solutions are $\sigma = -\frac{1}{3}$ and $\sigma = \frac{2}{3}$. These yield $\pi = \frac{1}{54}$ and $\pi = \frac{1}{9}$, so the answer is $\boxed{\frac{7}{54}}$.

23. Two points X and Y are chosen uniformly at random on the perimeter of equilateral $\triangle ABC$ with $AB = 10$. Given that the probability that $XY > 5$ is equal to $\frac{a}{b} - \frac{\pi}{\sqrt{c}}$ for positive integers a, b, and c with $\gcd(a, b) = 1$, find $a + b + c$.

Proposed by: Samuel Kretzschmar

Answer: $\boxed{250}$

Solution: There are two possible cases: either X and Y are on the same side with probability $\frac{1}{3}$, or on adjacent sides (as all sides of a triangle are adjacent) with probability $\frac{2}{3}$.

If they are on the same side, assume WLOG they are on AB. Then we can represent the two points by the lengths $AX = x$ and $AY = y$, which by uniform selection should be random values in the interval $[0, 10]$. Looking at the region $[0, 10]^2$ and shading the regions for which $XY = |x - y| > 5$ yields an area of $2 \cdot \frac{5^2}{2} = 25$, giving the probability $\frac{25}{100} = \frac{1}{4}$ for this case.

If they are on adjacent sides, then assume WLOG that X is on AB and Y is on AC. Using the same logic as the previous case, we set $AX = x$ and $AY = y$, noting that both are random values in the interval $[0, 10]$. We also note that by the Law of Cosines,

$$XY^2 = AX^2 + AY^2 - 2 \cdot AX \cdot AY \cos(60^\circ) = x^2 - xy + y^2,$$

so $XY > 5$ is equivalent to $x^2 - xy + y^2 > 25$. As an equation this represents an ellipse; we draw the ellipse and the sector of it in the region $[0, 10]^2$ in the diagrams below. (need to add) We want to find the area enclosed by both the ellipse and the region $[0, 10]^2$ (to take its complement), which is the same as the area of the ellipse in the first quadrant with $x, y > 0$.

We note that

$$x^2 - xy + y^2 = \frac{((x+y)/\sqrt{2})^2}{2} + \frac{((x-y)/\sqrt{2})^2}{2/3} = 25.$$

This means stretching out the ellipse by a factor of $\sqrt{3}$ along its minor axis forms it into a circle with radius $5\sqrt{2}$. The boundary of the original sector we want is bounded by the line $x = 0$, which makes a 45° angle with the major axis, having tangent of 1. When stretched, the minor axis (the opposite component) is multiplied by $\sqrt{3}$ while the major axis (the adjacent component) stays put, meaning the angle now has tangent $\sqrt{3}$, and is exactly 60°. The same is the case for $y = 0$, which when stretched forms another 60° angle with the minor axis. Therefore, when we stretch everything by $\sqrt{3}$ along the ellipse's minor axis, the elliptical sector becomes a circular sector of exactly 120°, with area $\dfrac{120^\circ}{360^\circ}\pi(5\sqrt{2})^2 = \dfrac{50\pi}{3}$, so the original elliptical sector had area $\frac{50\pi}{3\sqrt{3}}$. Thus the probability for this case is $1 - \dfrac{\left(\frac{50\pi}{3\sqrt{3}}\right)}{100} = 1 - \dfrac{\pi}{6\sqrt{3}}$.

Combining this result with the first case yields the final probability

$$\frac{1}{3} \cdot \frac{1}{4} + \frac{2}{3} \cdot \left(1 - \frac{\pi}{6\sqrt{3}}\right) = \frac{1+8}{12} - \frac{\pi}{9\sqrt{3}} = \frac{3}{4} - \frac{\pi}{\sqrt{243}}$$

for the final answer of $3 + 4 + 243 = \boxed{250}$.

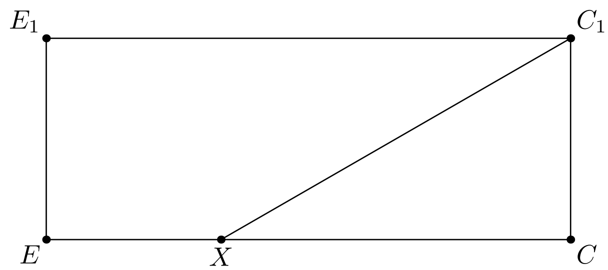
24. A right hexagonal prism has side length 6 and height of 4. The plane of the top face is rotated downwards by 30° around one of edges to form a new plane $\mathcal{P}$, which cuts the hexagonal prism into two parts. Compute the portion of $\mathcal{P}$ inside the prism.

Proposed by: Jack Gao

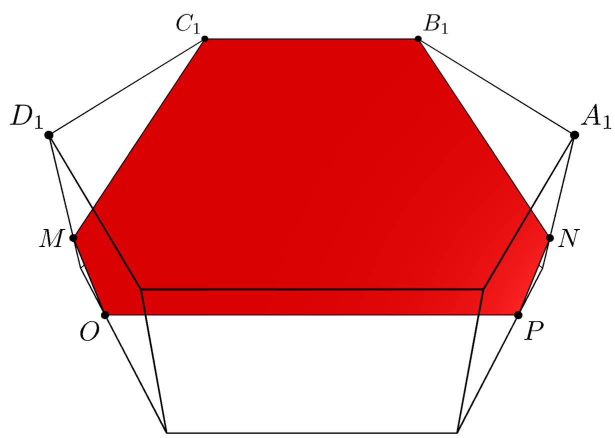
Answer: $\boxed{76}$

Solution: Let the bottom face of the hexagonal prism be $ABCDEF$ and the top face be $A_1B_1C_1D_1E_1F_1$, where A_1 is directly above A, and so on. Say that the plane (while we'll call $\mathcal{P}$ from now on) starts from segment B_1C_1.

We first need to figure out if $\mathcal{P}$ will intersect the prism again on the bottom face, or the face EFF_1E_1 (which is the opposite face to BCC_1B_1). Take a cross section of plane C_1CEE_1.



Let this plane, $\mathcal{P}$, and the plane of the bottom face intersect at X. Then because $\mathcal{P}$ has a 30° inclination, $\angle E_1C_1X = 30^\circ$. Also, ECC_1E_1 is a rectangle with $CC_1 = 4$ and $EC = 6\sqrt{3}$, so $\angle XC_1C = 60^\circ$ and now $C_1X = 8$ and $XC = 4\sqrt{3} < EC$. Thus, X is between E and C, so $\mathcal{P}$ meets the prism again at the bottom face.

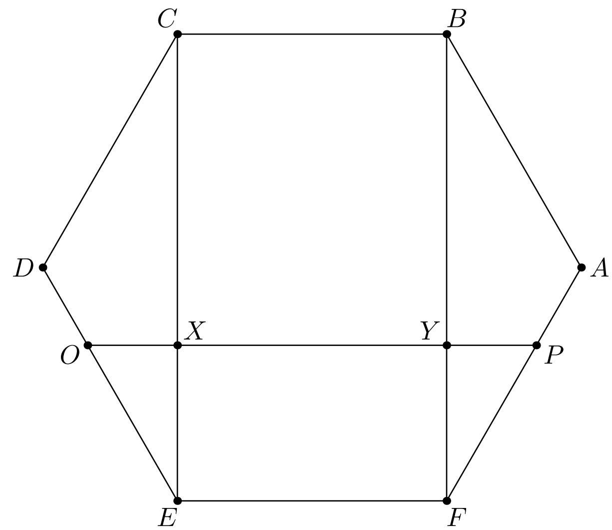


We omit the labels of A and D above to avoid clutter.

Now, let $\mathcal{P}$ intersect DD_1, AA_1, DE, and FA at M, N, O, and P, respectively. Then part of $\mathcal{P}$ that is cut out by the prism is the hexagon C_1MOPNB_1.

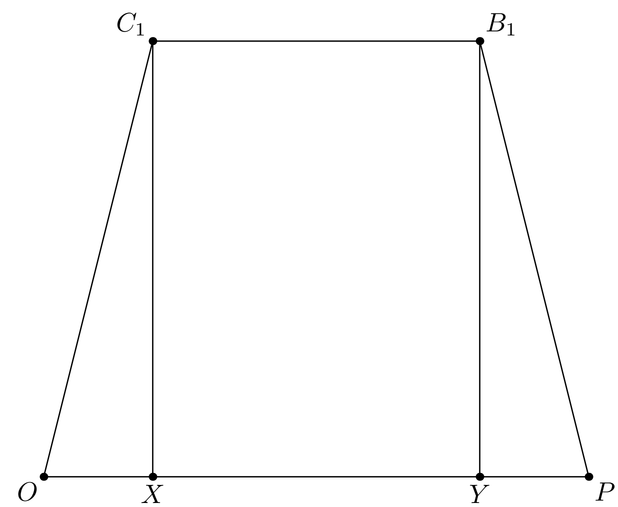
For symmetry reasons, C_1B_1PO is actually an isosceles trapezoid with $B_1C_1 \parallel OP$ and $OC_1 = B_1P$; also, $\triangle C_1MO \cong \triangle B_1NP$. Thus we will compute the area of the hexagon as $2[C_1MO] + [C_1B_1PO]$.

First, consider the plane of the bottom face.



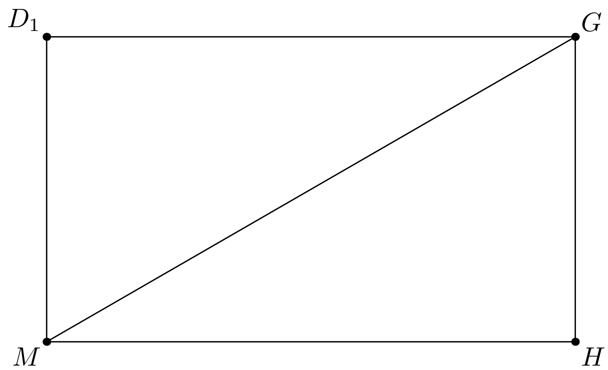
Line CE intersects OP at the same point X that we saw when taking the cross section in the plane of ECC_1E; thus, $CX = 4\sqrt{3}$. Similarly, if Y is the intersection of OP and BF, then $BY = 4\sqrt{3}$. Thus, $EX = FY = 2\sqrt{3}$, so since $\angle EOX = \angle EDA = 60° = \angle FPY$, we have $OX = PY = 2$ and $OE = PF = 4$.

Thus, $OD = PA = 2$, and $OP = OX + XY + YP = 2 + 6 + 2 = 10$.



In isosceles trapezoid B_1C_1OP, C_1X and B_1Y are the heights. Thus by pythag, $OC_1 = \sqrt{OX^2 + C_1X^2} = \sqrt{2^2 + 8^2} = 2\sqrt{17}$. Also, the area of B_1C_1OP is $\frac{1}{2}(6 + 10) \cdot 8 = 64$.

Now, take a plane parallel to plane ECC_1E_1 through M; let this intersect line B_1C_1 at a point G. Additionally, let H be the point so that MD_1GH is a rectangle.



Observe that MH is the distance from M to the plane BCC_1B, which is $\frac{1}{2} \cdot 6\sqrt{3} = 3\sqrt{3}$. Additionally, $\mathcal{P}$ is inclined 30° downwards from line B_1C_1, we have $\angle MGD_1 = 30^\circ$; thus, $MD_1 = \frac{D_1G}{\sqrt{3}} = \frac{3\sqrt{3}}{\sqrt{3}} = 3$. Therefore, $DM = 4 - MD_1 = 1$.

Thus, by pythag, $MC_1 = \sqrt{D_1M^2 + D_1C_1^2} = \sqrt{3^2 + 6^2} = 3\sqrt{5}$, and $MO = \sqrt{MD^2 + DO^2} = \sqrt{1^2 + 2^2} = \sqrt{5}$.

Now, MOC_1 and PNB_1 are both $3\sqrt{5}$–$\sqrt{5}$–$2\sqrt{17}$ triangles, meaning we can compute their areas by Heron's formula. Alternatively, we have by the law of cosines,

$$68 = OC_1^2 = OM^2 + MC_1^2 - 2OM \cdot MC_1 \cos(\angle OMC_1) = 50 - 30\cos(\angle OMC_1),$$

so $\cos(\angle OMC_1) = -\frac{3}{5}$, meaning $\sin(\angle OMC_1) = \frac{4}{5}$. Thus, $[OMC_1] = \frac{1}{2} \cdot OM \cdot MC_1 \sin(\angle OMC_1) = \frac{1}{2} \cdot 15 \cdot \frac{4}{5} = 6$.

Thus the area of the hexagon is

$$2[OMC_1] + [C_1B_1PO] = 12 + 64 = \boxed{76}.$$

25. The 2026 MN Youth Math Outreach staff team consists of Angie Huang, Michael Luo, Austin Wang, Jefferson Zhou, Taison Scofield, Zack Berchenko, Sneha Kundu, Samuel Kretzschmar, Eric Ding, Addison Shi, Andreas Walexon, Jack Gao, Allison Zhang, and Ansh Singh, who together attended some set of high schools in the 2025–26 school year. Estimate the area in square miles of the convex hull formed by these schools.

Submit a positive real number E. If the correct answer is A, you will earn $\lfloor 20.99 \cdot 2^{-16(1-E/A)^2} \rfloor$ points.

(The *convex hull* of a finite set of points $\mathcal{P}$ is the smallest convex polygon which contains all points in $\mathcal{P}$. It may be visualized as a rubber band stretched around $\mathcal{P}$.)

Proposed by: Michael Luo

Answer: $\boxed{1743.37}$

Solution: Angie, Michael, Austin, Jeff, Taison, Zack, Sneha, Sam, Eric, Addison, Andreas, Jack, Allison, and Ansh go to East Ridge, Minnetonka, Mounds View, Wayzata, Blaine, St. Paul Academy, Wayzata, Spring Lake Park, Century, Minnetonka, Minnetonka, St. Paul Academy, Sartell, and Century, respectively. The convex hull of these schools is shown in Figure 1.

The schools on the boundary of the convex hull are Minnetonka, Sartell, Blaine, East Ridge, and Century.

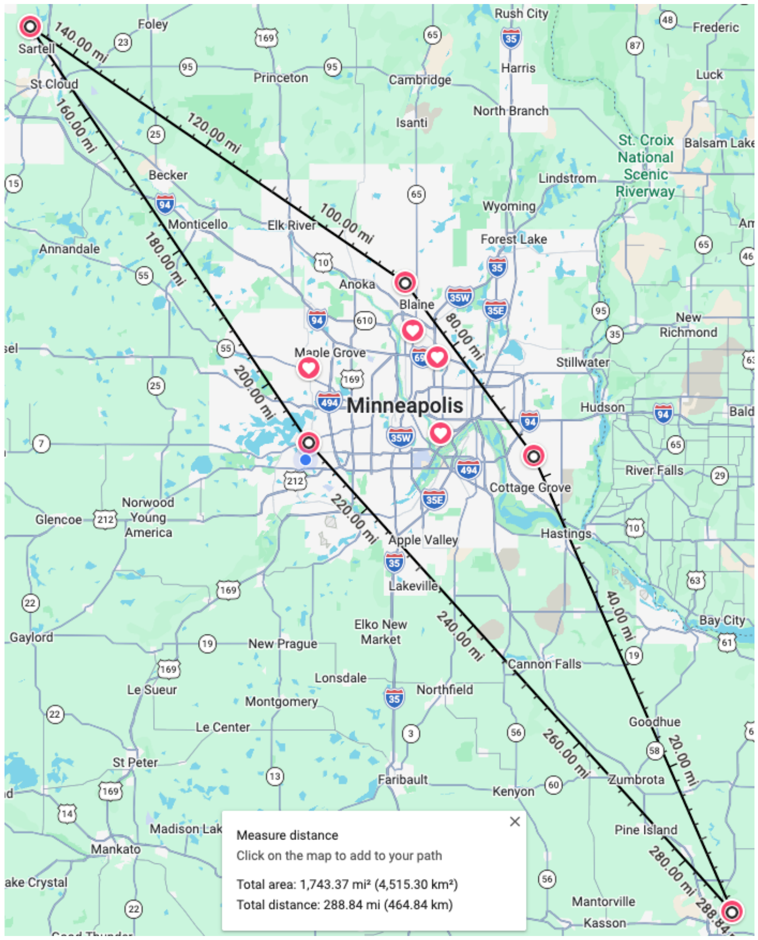


Figure 1: The schools in question with their convex hull.

26. Samuel Jesse Kretzschmar has recently turned 18. He found a very profitable slot machine at a casino, which is free to play and pays him 2^n dollars with probability $\frac{1}{2^n}$ for all integers $n \geq 1$. Estimate the expected number of times he must play before his profit first reaches 10^6 dollars.

Submit a positive real number E. If the correct answer is A, you will earn $\lfloor 20.99 \cdot 2^{-32(1-E/A)^2} \rfloor$ points.

Proposed by: Michael Luo

Answer: $\boxed{51277.00804003184}$

Solution: Let a_n be the expected time required to earn $2n$ dollars. We have $a_n = 0$ for $n \leq 0$. As the probability of earning 2^n dollars is $\frac{1}{2^n}$ for all $n \geq 1$, we have the recurrence relation

$$a_n = 1 + \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} a_{n-2^i}.$$

Let $N = \frac{1}{2} \cdot 10^6$; we want a_N. Near N, we may approximate a_n linearly as αn for some α.

Plugging into the recurrence yields

$$\alpha n = 1 + \sum_{i=0}^{\lfloor \log_2 n \rfloor} \alpha \frac{n - 2^i}{2^{i+1}},$$

or equivalently,

$$n = \frac{1}{\alpha} + n\left(1 - \frac{1}{2^{\lfloor \log_2 n \rfloor}}\right) - \frac{1}{2}(\lfloor \log_2 n \rfloor + 1),$$

so

$$\alpha = \frac{1}{\frac{1}{2}(\lfloor \log_2 n \rfloor + 1) + \frac{n}{2^{\lfloor \log_2 n \rfloor}}}.$$

As $n \approx N$, we get $\alpha \approx \frac{2}{23}$ and $N \approx 43478$, earning 12 points.

Using our original recursion and Python, we may compute the exact answer

$$a_N \approx \boxed{51277.00804003184}.$$

```python
from math import log2

a = [0]
for i in range(1,500001):
    temp = 1
    for j in range(int(log2(i) + 1)):
        temp += a[-2 ** j] / 2 ** (j + 1)
    a.append(temp)

print(a[-1])
```

27. A prime is *mnymoy* if it can be written as the concatenation of n and $n+1$ (in that order) for some positive integer n such that either n or $n+1$ is prime. For example, 67 is mnymoy, since it is the concatenation of 6 and 7, and 7 is prime. Compute the number of mnymoy integers at most 10^{10}.

Submit a positive real number E. If the correct answer is A, you will earn $\lfloor 20.99 \cdot 2^{-32(1-E/A)^2} \rfloor$ points.

Proposed by: Samuel Kretzschmar and Michael Luo

Answer: $\boxed{896}$

Solution: Throughout this solution, let log denote the natural logarithm. We claim that if the concatenation of n and $n+1$ is prime, then $n+1$ is prime. If $n = 2$, this is clear, so suppose $n \geq 3$. If the concatenation of n and $n+1$ is mnymoy, it is prime and hence odd, so $n+1$ is odd, so n is even and at least 3. Therefore, n is not prime, so $n+1$ is prime. Hence it's equivalent to need $n+1$ prime and the concatenation of n and $n+1$ prime.

Let $N = 10^5$. For a random $n \leq N$, the "probability" that $n+1$ is prime is $\frac{1}{\log N}$ and the "probability" that the concatenation of n and $n+1$ is prime is $\frac{1}{\log(N^2)}$, by the prime number theorem. However, since the last digit of $n+1$ and the concatenation of n and $n+1$ is shared, given $n+1$ is prime, the concatenation of n and $n+1$ is around $\frac{10}{4}$ times more likely to be prime, since the last digit of $n+1$ is 0, 2, 4, 5, 6, or 8 only once. Hence we estimate $N \cdot \frac{1}{\log N} \cdot \frac{1}{\log(N^2)} \cdot \frac{10}{4} \approx 1000$, earning 15 points.

The exact answer $\boxed{896}$ may be computed using the following Python code.

```
from sympy import isprime
print(sum((isprime(k) or isprime(k + 1)) and isprime(int(str(k) + str(k + 1))) for
    k in range(10**5)))
```