MNYMO
August 22, 2026
Mastery

The answer key is below.

p	Answer	p	Answer
1	1	5	$\frac{\pi+4-2\sqrt{2}}{4}$
2	73	6	-32
3	$\frac{2}{11}$	7	$\frac{16}{27}$
4	$\frac{576}{5}$	8	433

1. Jeff has a real number L for which

$$L + \phi = \phi^2,$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. What is L?

Proposed by: Jefferson Zhou

Answer: $\boxed{1}$

Solution: We have

$$L = \left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1+\sqrt{5}}{2}\right) = \frac{3+\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2} = \boxed{1}.$$

Alternative Solution: Note that ϕ is the positive solution to the quadratic $x^2 - x - 1 = 0$, which means that $\phi^2 - \phi - 1 = 0$, so $L = \boxed{1}$.

2. Austin is trying to remember his favorite basketball player's jersey number, which is an integer n between 0 and 99, inclusive. However, he can only remember the following information about it:

   a) n leaves a remainder of 1 when divided by 4.

   b) n leaves a remainder of 3 when divided by 5.

   c) n leaves a remainder of 3 when divided by 7.

What is n?

Proposed by: Jack Gao

Answer: $\boxed{73}$

Solution: We can test out numbers that satisfy the first condition: 5, 9, 13, and so on, and we can see that 13 satisfies the first two conditions. Now, since $\operatorname{lcm}(4, 5) = 20$, we start adding by 20s: 13, 33, 53, 73, and so on. We can see that $\boxed{73}$ satisfies all three congruencies.

Alternative Solution: The last two pieces of information are equivalent together to $n \equiv 3 \pmod{35}$ by CRT. However, 3 is 3 modulo 4, so we need to add $2 \cdot 35$ to get a number equivalent to 0 modulo 4, meaning $n = 73$ works. The three congruences are now equivalent to $n \equiv 73 \pmod{140}$, and $\boxed{73}$ is indeed the only solution.

3. Jeff is trying to arrive to lunch at precisely noon. However, because he is terrible with times, he will arrive uniformly at random between 11:30 AM and 12:30 PM. What is the probability that, when he arrives at lunch, the smaller angle between the hour and minute hands of a clock set to the local time is less than $30°$?

Proposed by: Jefferson Zhou

Answer: $\boxed{\frac{2}{11}}$

Solution: The minute hand travels at $\frac{360}{60} = 6$ degrees per minute. The hour hand travels at one-twelfth that speed, or $\frac{1}{2}$ degrees per minute. Thus the minute hand gains on the hour hand by $6 - \frac{1}{2} = \frac{11}{2}$ degrees per minute.

At precisely noon, the hour and minute hands have the same direction. Thus, 30 minutes later at 12:30 PM, the minute hand has gained on the hour hand by $\frac{11}{2} \cdot 30 = 165$ degrees. Similarly, 30 minutes earlier, at 11:30 AM, the minute hand is behind the hour hand by $\frac{11}{2} \cdot 30 = 165$ degrees.

Thus, during the possible time period, the minute hand gains on the hour hand by a total of 330 degrees. The angle between the hands is less than 30° from the time when the hour hand is ahead of the minute hand by 30 degrees to the time when the minute hand is ahead of the hour hand by 30 degrees, during which the minute hand gains on the hour hand by a total of 60 degrees. Thus the angle between the hands is less than 30° for $\frac{60}{330} = \boxed{\frac{2}{11}}$ of the total time period.

4. Let ω be a circle of radius 6 centered at a point A, and let point B lie on ω. Point C lies outside ω so that BC is tangent to ω. Point D is the intersection of ω with AC such that A is between C and D. Given that $\frac{BC}{AC} = \frac{4}{5}$, determine BD^2.

   Proposed by: Jack Gao

   Answer: $\boxed{\frac{576}{5}}$

   Solution: Since BC is tangent to ω at B, we have $\angle ABC = 90°$. By $\frac{BC}{AC} = \frac{4}{5}$, $\triangle ABC$ is a 3–4–5 right triangle. Thus, since $AB = 6$, $BC = 8$ and $AC = 10$.

   Note that

$$\cos(\angle BAD) = \cos(180° - \angle BAC) = -\cos(\angle BAC) = -\frac{AB}{AC} = -\frac{3}{5}.$$

   Now, using the Law of Cosines,

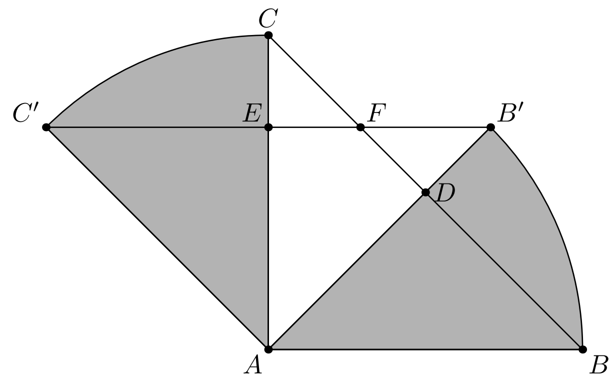
$$\begin{aligned} BD^2 &= AD^2 + AB^2 - 2 \cdot AB \cdot AD \cdot \cos(\angle BAD) \\ &= 6^2 + 6^2 - 2 \cdot 6 \cdot 6 \cdot \left(-\frac{3}{5}\right) \\ &= 72 + \frac{216}{5} \\ &= \boxed{\frac{576}{5}} \end{aligned}$$

5. A 45–45–90 right triangle has legs of length 1. The triangle is rotated 45° about its right-angle vertex. What is the area of the region swept out by the triangle?

   Proposed by: Taison Scofield

   Answer: $\boxed{\frac{\pi+4-2\sqrt{2}}{4}}$

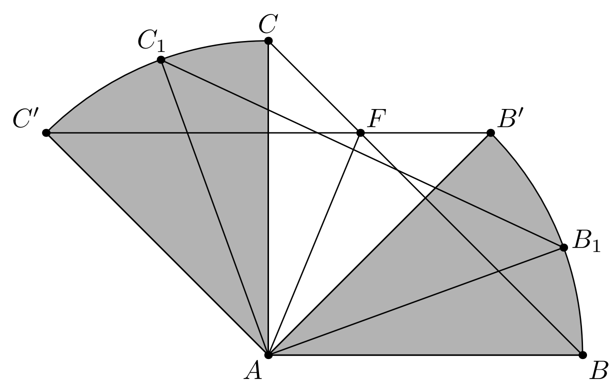
   Solution: Let $\triangle ABC$ be the original triangle and let $A'B'C'$ be the final image. Additionally, let BC intersect AB' and $B'C'$ at D and F, respectively, and let AC intersect $B'C'$ at E.



Observe that as we rotate, segments AB and AC both sweep out one-eighth of a circle centered at A, as shown above. Thus it remains to determine what region is swept out in the middle of these two.

We claim that, in fact, the desired region is just the concave quadrilateral $ACFB'$.

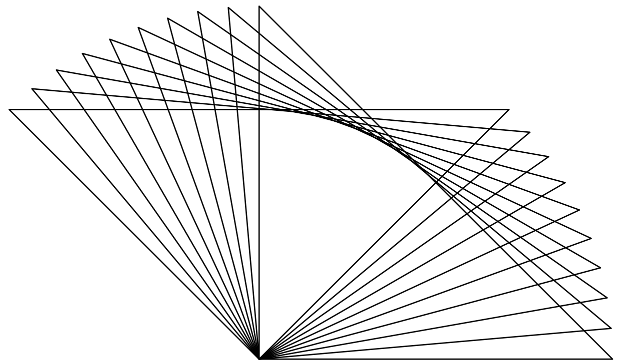
Consider a moving intermediate state $A_1B_1C_1$ of the triangle, as shown below.



Observe that the part of $\triangle A_1B_1C_1$ that is not in either sector but lies below AF always lies below line $B'F$, because the line B_1C_1 is always of negative slope (if AB and AC are the x- and y- axes, respectively) and always intersects line AF between A and F (as if B_1C_1 intersects AF at a moving point P, then the distance AP decreases from AF down to the minimum when $AF \perp B_1C_1$, and then rises again until it becomes AF again at the end). By identical reasoning (as the entire figure is symmetrical across AF), the part of $\triangle A_1B_1C_1$ that lies above AF but is not in any sector lies below line FC; therefore, the part of $\triangle A_1B_1C_1$ not in any sector always lies entirely within the quadrilateral $AB'FC$. Additionally, we can see that the starting and ending states cover the entirety of this quadrilateral, so this is the desired region swept out by the triangle between the sectors.

Now, if we go back to the first diagram, we just want to compute the area of the entire bounded region. The two sectors combined comprise a fourth of a circle of radius 1, or an area of $\frac{\pi}{4}$. Note that $AE = \frac{AC'}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, so $FD = FE = CE = 1 - \frac{\sqrt{2}}{2}$; therefore, $[AB'FC] = [AB'F] + [ACF] = 2 \cdot \frac{1}{2} \cdot \left(1 - \frac{\sqrt{2}}{2}\right) = \frac{2-\sqrt{2}}{2}$. Thus the final answer is $\frac{\pi}{4} + \frac{2-\sqrt{2}}{2} = \boxed{\frac{\pi+4-2\sqrt{2}}{4}}$.

For reference, below is an image of the triangle rotated every 5 degrees up to the total 45 degrees. The desired region is apparent.



6. Let a and b be the roots of the quadratic equation $x^2 - 20x + 26 = 0$. Let $f(x) = px^2 + qx + r$ be a quadratic function such that $f(a) = b$, $f(b) = a$, and $f(1) = 5$. Find the value of $f(0)$.

   Proposed by: Taison Scofield

   Answer: $\boxed{-32}$

   Solution: By Vieta's, $a + b = 20$ and $ab = 26$. The equations become

$$\begin{aligned} pa^2 + qa + r &= b \\ pb^2 + qb + r &= a. \\ p + q + r &= 5 \end{aligned}$$

   Adding the first two, we get

$$p(a^2 + b^2) + q(a + b) + 2r = p(20^2 - 2 \cdot 26) + 20q + 2r = 348p + 20q + 2r = a + b = 20,$$

   so $174p + 10q + r = 10$, meaning $173p + 9q = 5$.

   Subtracting the first two, we get $p(a^2 - b^2) + q(a - b) = b - a$, so dividing by $a - b$ yields $p(a + b) + q = 20p + q = -1$.

   Now, $7p = 9(20p + q) - (173p + 9q) = -14$, so $p = -2$, so $q = -1 - 20p = 39$, so $f(0) = r = 5 - p - q = \boxed{-32}$.

   Alternative Solution: Let $g(x) = f(x) + x - 20$. Note that because $a + b = 20$, we have $g(a) = f(a) + a - 20 = a + b - 20 = 0 = g(b)$, so g has roots a and b, meaning it is a multiple of $x^2 - 20x + 26$. Thus, there exists a constant c for which $c(x^2 - 20x + 26) = f(x) + x - 20$. Setting $x = 1$ yields $7c = 5 + 1 - 20 = -14$, so $c = -2$. Now, $f(0) = 20 - 0 - 2 \cdot 26 = \boxed{-32}$.

7. Alice and Bob are playing a game with a penny. Alice starts by flipping the penny; if it lands heads, she flips it again, but if it's tails, she gives the coin to Bob. She continues until she flips tails, at which point Bob does the same process, giving the penny to Alice once he flips tails, and so on indefinitely. What is the probability that Alice flips two heads (total) first?

   Proposed by: Samuel Kretzschmar

   Answer: $\boxed{\frac{16}{27}}$

   Solution: For $x, y \leq 1$, let $p_{x,y}$ be the probability that Alice wins assuming that Alice starts with x heads and Bob starts with y heads.

If Alice gets the $\frac{1}{4}$ chance of just directly getting two heads, she immediately wins. If Alice gets one head and one tail (probability $\frac{1}{4}$), then to Bob, it is like he started first, but the opponent already has 1 head; thus, in this case, Bob has a $p_{0,1}$ probability of winning, so Alice has a $1-p_{0,1}$ chance of winning. Similarly, if Alice gets the $\frac{1}{2}$ chance of getting no heads, the probability that she wins is $1-p_{0,0}$. Thus

$$p_{0,0} = \frac{1-p_{0,0}}{2} + \frac{1-p_{0,1}}{4} + \frac{1}{4},$$

which simplifies to $\frac{3p_{0,0}}{2} = 1 - \frac{p_{0,1}}{4}$.

We can do the same analysis to find an equation for $p_{0,1}$; if Alice gets two heads, she wins, if Alice gets one head, she wins with probability $1-p_{1,1}$, and if she gets no heads, she wins with probability $1-p_{1,0}$. Thus,

$$p_{0,1} = \frac{1-p_{1,0}}{2} + \frac{1-p_{1,1}}{4} + \frac{1}{4},$$

or $p_{0,1} + \frac{p_{1,0}}{2} = 1 - \frac{p_{1,1}}{4}$.

We now do the same thing for $p_{1,0}$; If Alice gets one head, she wins, and if she gets no heads, she wins with probability $1-p_{0,1}$. Thus,

$$p_{1,0} = \frac{1-p_{0,1}}{2} + \frac{1}{2},$$

or $p_{1,0} + \frac{p_{0,1}}{2} = 1$. Summing the last two yields $\frac{3}{2}(p_{0,1}+p_{1,0}) = 2 - \frac{p_{1,1}}{4}$, or $p_{0,1}+p_{1,0} = \frac{4}{3} - \frac{p_{1,1}}{6}$.

Repeating for $p_{1,1}$, if Alice gets 1 head, she wins, and if she gets no heads, then she has a $\frac{1-p_{1,1}}{2}$ chance of winning, so

$$p_{1,1} = \frac{1-p_{1,1}}{2} + \frac{1}{2},$$

or $\frac{3p_{1,1}}{2} = 1$, so $p_{1,1} = \frac{2}{3}$.

Now, $p_{0,1}+p_{1,0} = \frac{4}{3} - \frac{1}{9} = \frac{11}{9}$, so $\frac{11}{9} = p_{1,0}+p_{0,1} = 1 + \frac{p_{0,1}}{2}$ and $p_{0,1} = \frac{4}{9}$, so $p_{0,0} = \frac{2}{3}\left(1 - \frac{p_{0,1}}{4}\right) = \boxed{\frac{16}{27}}$.

8. For all positive integers n, let $\tau(n)$ denote the number of positive divisors of n. Compute the sum of all positive integers n for which the nine numbers

$$\tau(n), \tau(n+2), \tau(n+4), \ldots, \tau(n+16)$$

are all less than 6.

Proposed by: Jefferson Zhou

Answer: $\boxed{433}$

Solution: Observe that the nine numbers n, $n+2$, $n+4$, ..., $n+16$ are all distinct modulo 9, so one of them is divisible by 9. If a number divisible by 9 is divisible by some prime number other than 3, than that number has at least $3 \cdot 2 = 6$ divisors. Thus, this number must be a power of 3, and the only possibilities are $3^2 = 9$, $3^3 = 27$, or $3^4 = 81$. In particular, this means n is odd.

We can manually check that all odd $n \leq 43$ satisfy $\tau(n) < 6$, meaning all odd $1 \leq n \leq 27$ work; this covers the cases where the multiple of 9 is 9 or 27. The solutions here sum to $1 + 3 + \cdots + 27 = 14^2 = 196$.

We just need to check the cases where the multiple of 9 is 81. We can first note that $\tau(75) = 6$, so any solution n in that case needs to be at least 77; at the same time, we must have $n \leq 81$ to make sure $81 \in \{n, n+2, \ldots, n+16\}$. However, we can compute

$$\begin{aligned}
\tau(77) &= 4 \\
\tau(79) &= 2 \\
\tau(81) &= 5 \\
\tau(83) &= 2 \\
\tau(85) &= 4 \\
\tau(87) &= 4 \\
\tau(89) &= 2 \\
\tau(91) &= 4 \\
\tau(93) &= 4 \\
\tau(95) &= 4 \\
\tau(97) &= 2,
\end{aligned}$$

which are all less than 6, so $n = 77$, 79, 81 all work. These sum to $3 \cdot 79 = 237$.

There are no more solutions, so the final answer is $196 + 237 = \boxed{433}$.